From Interacting Husserlian Monads to Jordan State Spaces

Recursive Constitution, Modal Completeness, Kinematic Homogeneity, and the Reconstruction of Quantum-Like Geometry

A self-contained research synthesis in the monadological program

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Abstract

This paper reconstructs and audits an exploratory program that begins with interacting Husserlian monads and asks how much of quantum state-space geometry can be recovered from principles stated in phenomenological, constitutional, and operational terms. The development begins with a minimal three-monad actual-world representation problem, passes from classical to quantum and generalized probabilistic models, and studies principles intended to exclude Popescu–Rohrlich-type superquantum correlations. Information-causality-like constraints recover the Tsirelson bound on an isotropic slice but do not characterize the full quantum correlator set. A stronger route through Euclidean Jordan algebras (EJAs) yields the Tsirelson–Landau–Masanes correlator inequality once a state-dependent inner product and a product rule for jointly sharp conjunctions are available. Several initially attractive claims are then rejected or weakened: generic pure entanglement does not induce isometric steering; purification is too strong to serve as an unanalysed primitive; recursive articulation alone does not imply symmetry; and kinematic completeness does not by itself imply dynamical richness.

The paper develops a replacement architecture. Recursive Constitutive Articulation (RCA) supplies spectrality. Hereditary Symmetry of Standing (HSS) is identified as a recursive stabilizer form of strong symmetry. Its kinematic and dynamical components are separated into No Ungrounded Kinematic Asymmetry (NUKA) and No Ungrounded Dynamical Asymmetry (NUDA). NUKA is further analyzed as a one-point homogeneity property that can, in principle, arise from Context-Grounded Constitutive Amalgamation and Horizon Saturation. These two principles are shown independent by explicit countermodels and are unified by a stronger Contextual Constitutive Modal Completeness (CCMC) principle: every finitely lawful contextual extension pattern is internally realized without ungrounded identifications. A deliberately permissive Pre-Jordan Admissibility Kernel (PJAK) is then proposed, together with a metric constitutive language $\mathcal{L}_{\text{mon}}$ of states, effects, atomic sharp outcomes, contexts, convex operations, refinement, and grounded background predicates. Finally, a provisional Monadic Jordan Reconstruction Theorem is formulated: under PJAK, pure-sharp completeness, sectorial frame anonymity, Peeling RCA, and a metric Fraisse-style form of CCMC, the compact finite-dimensional state space is spectral and strongly symmetric; by the theorem of Barnum and Hilgert, it is therefore affinely isomorphic to a simplex or the normalized state space of a simple EJA. NUDA is not needed for this kinematic Jordan conclusion, but upgrades it to physical reversible strong symmetry. The result is a compact research program in which Jordan geometry is no longer postulated through self-duality, homogeneity, purification, or reversible filters, but is targeted as a consequence of recursive constitution plus modal completeness.

Contents

1 Status, Scope, and Antecedents

The present paper is a synthesis of a long exploratory dialogue rather than a report of a finished axiomatization. It deliberately distinguishes four epistemic levels.

1. **Published mathematical results.** These include the Tsirelson bound, the Tsirelson–Landau–Masanes characterization of the unbiased binary correlator region, information causality, standard purification results in generalized probabilistic theories (GPTs), Wilce's conjugate/filter and subspace reconstructions, the Barnum–Hilgert characterization of spectral strongly symmetric convex bodies, strong Boolean amalgamation for orthomodular lattices, and metric Fraisse theory.
2. **Elementary consequences of definitions introduced here.** Examples include the equivalence between Hereditary Symmetry of Standing and strong symmetry on finite distinguishable lists, the inductive proof that Peeling RCA implies spectrality, and explicit independence countermodels for amalgamation and horizon saturation.
3. **Conditional synthesis results.** Examples include the proposed route from reciprocal horizon completion to weak self-duality, the derivation of a TLM-type Gram representation from an EJA product rule, and the provisional Monadic Jordan Reconstruction Theorem. These are mathematically motivated but require a fully formal category/language before they should be treated as established theorems.
4. **Phenomenological bridge principles.** Claims such as "no ungrounded modal exclusion" or "no ungrounded dynamical asymmetry" are proposed foundational readings of mathematical assumptions, not consequences of Husserl's phenomenology alone.

The project extends earlier experimental manuscripts in which Husserlian monads were represented as finite intentional systems and assembled into interacting worlds, diachronic overminds, self-conscious overminds, and later theories of wanting and self-authorship. The two principal antecedent notes were *Possible Worlds of Interacting Husserlian Monads* and *Minimal Finite Models of Diachronic and Self-Conscious Overminds*; subsequent notes developed proleptic wanting, internal constitutions, self-authorship, and monadological coherent extrapolated volition. These antecedents motivate the language of noemata, horizons, retention, protention, standing, and constitutional refinement used below. They are not assumed by the technical convex-geometry results; the present paper is self-contained at the level required for the reconstruction.

The central question is an "actual-world representation problem": can a world built from interacting perspectival subjects possess an objective operational structure rich enough to reproduce the characteristic geometry and nonlocal correlations of quantum theory without simply inserting Hilbert space at the start?

2 The Actual-World Representation Problem

2.1 Formal monads

A minimal formal Husserlian monad is represented schematically as

$$\mathcal{M}_i = (X_i, N_i, R_i, P_i, A_i),$$

where X_i is an internal state space, N_i a noematic presentation map or structure, R_i retention, P_i protention, and A_i a family of acts or transformations. The intended reading is not that a subject literally is a five-tuple, but that the tuple carries enough structural roles to represent perspectival givenness, temporal synthesis, anticipatory horizon, and active constitution.

An objective object is not initially taken to be a God's-eye bearer of pre-existing values. It is instead represented by stable equivalence or descent structure across interacting perspectives. This motivates moving quickly from deterministic hidden variables to operational theories of contextual appearances.

2.2 The classical three-monad world

The simplest nontrivial world used in the dialogue was

$$W_3 = (A, B, X),$$

with two observer monads A, B and one mediator/source X. A binary mediator state S_t produces stochastic local appearances. Under the constraints that (i) there are two genuinely distinct observers, (ii) the object-support process is distinct from both observers, (iii) the observers do not directly communicate during the relevant measurement stage, and (iv) the mediator has nontrivial stochastic state, three monads are minimal.

This model can represent intersubjective objectivity as compatibility among appearances, but it remains a local hidden-variable world. Consequently its binary correlators obey Bell inequalities. The first lesson is therefore negative: a purely classical mediator between monads cannot reproduce the actual quantum pattern of Bell violation.

2.3 The quantum three-monad world

Replace the mediator by a bipartite quantum relation. In the binary-input/binary-output CHSH scenario, take

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

with

$$A_0 = \sigma_z, \qquad A_1 = \sigma_x,$$

$$B_0 = \frac{\sigma_z + \sigma_x}{\sqrt{2}}, \qquad B_1 = \frac{\sigma_z - \sigma_x}{\sqrt{2}}.$$

Then

$$S := E_{00} + E_{01} + E_{10} - E_{11} = 2\sqrt{2},$$

the Tsirelson value [5]. In the monadological interpretation, the "object" is no longer a global value assignment but a compatible family of contextual appearances together with relational structure constraining their joint statistics.

2.4 The generalized probabilistic hierarchy

To avoid assuming quantum theory, the dialogue next moved to GPTs. For a binary scenario one may write

$$p(a, b \mid x, y) = \frac{1}{4}\left(1 + a\alpha_x + b\beta_y + abE_{xy}\right),$$

where $a, b \in \{\pm 1\}$. The familiar hierarchy is

$$\mathsf{L} \subsetneq \mathsf{Q} \subsetneq \mathsf{NS},$$

where L is the local polytope, Q the quantum set, and NS the no-signalling polytope. PR correlations achieve $|S| = 4$ while respecting no-signalling [6, 7].

The reconstruction problem therefore sharpens: what monadologically intelligible principle separates Q from the larger no-signalling world?

3 First Attempts to Exclude Superquantum Correlations

3.1 Conditional Constitution and continuous reversible refinement

Two early principles were proposed. *Conditional Constitution* says that conditioning on an admissible outcome produces another admissible state. *Continuous Reversible Refinement* (CRR) says, roughly, that equally informative pure refinements are connected by continuous reversible transformations.

A PR-box local state space is naturally represented by a square gbit. Its reversible automorphism group is discrete (the dihedral symmetries of the square), so there is no continuous group connecting all pure extremal directions. This makes PR-type models unattractive under CRR. However, CRR by itself was not developed into a full characterization of Q.

3.2 Bounded Communicative Fulfillment and information causality

A second route translated information causality into monadological language. Let Alice hold random bits $r_1, \ldots, r_N$, communicate m classical bits, and let Bob produce guesses G_k. The proposed Bounded Communicative Fulfillment principle (BCF) is

$$\sum_{k=1}^{N} I(r_k : G_k) \leq m.$$

This is the information-causality inequality introduced by Pawlowski et al. [10], here interpreted as a limit on how much intentional/communicative fulfillment can be unlocked by m classical bits.

For the isotropic unbiased correlator family

$$(E_{00}, E_{01}, E_{10}, E_{11}) = (v, v, v, -v),$$

concatenated random-access-code protocols give

$$I_n = 2^n \left[1 - h_2\left(\frac{1+v^n}{2}\right)\right].$$

Using the quadratic lower bound near 1/2,

$$1 - h_2\left(\frac{1+t}{2}\right) \geq \frac{t^2}{2\ln 2},$$

one obtains

$$I_n \geq \frac{(2v^2)^n}{2\ln 2}.$$

If BCF is to hold for arbitrarily large concatenation depth, then $2v^2 \leq 1$, hence

$$v \leq \frac{1}{\sqrt{2}}, \qquad |S| = 4v \leq 2\sqrt{2}.$$

Thus the Tsirelson bound appears on this isotropic slice.

3.3 Why information causality is not enough

The dialogue then checked a crucial limitation. Information-causality-type quadratic constraints do not characterize the full two-input/two-output quantum correlator region. An explicit unbiased correlator such as

$$E_{00} = 1, \qquad E_{01} = E_{10} = 0.55, \qquad E_{11} = -0.55$$

can satisfy the simple Uffink/information-causality quadratic test while violating a Tsirelson–Landau–Masanes (TLM) condition. Masanes gives a necessary and sufficient nonlinear characterization for bipartite correlators with two dichotomic observables per party [8, 9]. The lesson was decisive:

$$\boxed{\text{Recovering the CHSH radius is weaker than recovering quantum correlator geometry.}}$$

This motivated a move from communication inequalities to state-space geometry.

4 The EJA Route to TLM Geometry

4.1 A false start: generic pure steering is not isometric

An early proposal claimed that a generic pure bipartite state should induce an isometric steering correspondence between local effect spaces. This is false even in ordinary quantum theory. For

$$|\psi_\lambda\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle,$$

the two Schmidt directions are weighted unequally. Transverse Bloch components are distorted by a factor proportional to

$$2\sqrt{\lambda(1-\lambda)},$$

which equals one only at maximal entanglement. The correction is conceptually important: generic purity is not the same as canonical reciprocity or maximal entanglement.

4.2 State-dependent inner products in an EJA

In a Euclidean Jordan algebra J, a state ρ induces the bilinear form

$$(r, s)_\rho := \langle \rho, r \circ s\rangle,$$

where $\circ$ is the Jordan product. For binary sharp contrasts A_x, B_y of norm at most one, if local conjunction is represented by the Jordan product, then

$$E_{xy} = \langle \rho, A_x \circ B_y\rangle = (A_x, B_y)_\rho.$$

After quotienting null directions and normalizing, the correlators therefore admit a Gram representation by unit vectors. This is exactly the geometry underlying the TLM inequalities.

One convenient Landau form is

$$|E_{00}E_{10} - E_{01}E_{11}| \le \sqrt{(1-E_{00}^2)(1-E_{10}^2)} + \sqrt{(1-E_{01}^2)(1-E_{11}^2)}.$$

For unbiased 2×2 correlators this is part of the necessary-and-sufficient quantum characterization associated with Tsirelson, Landau, and Masanes [5, 8, 9].

4.3 Reciprocal Conjunctive Constitution

The product rule was named *Reciprocal Conjunctive Constitution* (RCC). A sharper operational source is *Joint Sharp Refinability* (JSR): if two compatible binary sharp observables admit a common atomic refinement $\{r_{ab}\}$, define

$$p_a = \sum_b r_{ab}, \qquad q_b = \sum_a r_{ab}.$$

In a Jordan frame, orthogonal idempotency gives

$$p_a \circ q_b = r_{ab}.$$

Thus RCC is not an arbitrary product postulate once one already has an EJA and sufficiently sharp joint refinement.

The dialogue further checked that arbitrary binary effects, not only atomic contrasts, can be spectrally decomposed in Jordan frames; under a suitable composite rule their cross-system products inherit the same norm bounds. This motivated the provisional synthesis that finite-dimensional sharp theories with purification should place unbiased binary correlators within the TLM quantum region. This statement should be treated as a theorem candidate contingent on the precise composite assumptions, not as a published theorem in this exact form.

5 Auditing Purity Preservation and Purification

5.1 Splitting Purity Preservation

Purity Preservation was decomposed into

$$PP_\circ : \quad \text{sequential composition of pure transformations is pure,}$$

$$PP_\otimes : \quad \text{parallel composition of pure transformations is pure.}$$

A proposed "Pure Local Conditioning" principle turned out, together with purification and $PP_\circ$, to reconstruct much of $PP_\otimes$, so it was not a genuine weakening.

The discussion then isolated two more elementary state-level principles:

Pure Conditioning (PC). Conditioning a pure bipartite state by a pure/atomic effect gives either zero or a pure conditional state.

Independent Pure Composition (IPC). If α and β are pure, then $\alpha \otimes \beta$ is pure.

Using standard purification/steering and link-product machinery inspired by Chiribella, D'Ariano and Perinotti [11], the dialogue argued conditionally that

$$P + PC + IPC \Longrightarrow PP_\circ + PP_\otimes.$$

This implication is a synthesis result whose fully formal proof depends on the exact process category.

5.2 Pure Intersubjective Extensionality and Constitutional Separatedness

IPC was then grounded in a more structural principle. *Pure Intersubjective Extensionality* (PIE) says that if a pure bipartite whole has exactly the product statistics of fixed pure local relata, then it is the product state.

The cleaner formulation is *Constitutional Separatedness* (also called Pure-Determination Separatedness in the working dialogue): for pure global states Ψ, Φ with the same pure local marginals α, β,

$$\Psi_A = \Phi_A = \alpha, \qquad \Psi_B = \Phi_B = \beta \quad \Longrightarrow \quad \Psi = \Phi.$$

Equivalently, the fiber of pure wholes over a pure-pure marginal pair has cardinality at most one.

A simple lemma supports the equivalence with PIE: if a bipartite state has a pure marginal α, every conditional state on that side must be proportional to α; hence product statistics factor. From Constitutional Separatedness one can derive IPC: if $\alpha \otimes \beta$ decomposed nontrivially into pure branches, every branch would inherit the same pure marginals, so separatedness forces all branches to coincide.

The philosophical content is "no pure relational haecceity over completely fixed pure relata." Importantly, this is weaker than local tomography and does not forbid entanglement when marginals are mixed.

A toy countertheory showed why causal independence/no-signalling alone does not imply IPC: two distinct pure global states Γ_0, Γ_1 can share the same pure marginals while the stipulated bare product is their mixture.

5.3 Why full Purification was judged too heavy

Purification consists of:

1. existence: every mixed state is a marginal of a pure bipartite state;
2. essential uniqueness: two purifications differ by a reversible transformation on the purifying system.

Chiribella et al. show that purification is equivalent, in their operational framework, to reversible realization of every physical process and yields a Choi-like state-transformation correspondence [11]. This power is exactly why the dialogue became reluctant to treat purification as a primitive monadological truth. Classical probability fails ordinary purification, while much weaker extension principles allow PR/boxworld correlations. A lighter reciprocity principle was sought instead.

6 Reciprocal Horizon Completion

6.1 Definition and steering map

Reciprocal Horizon Completion (RHC) postulates that each system A has a reciprocal/conjugate $\bar{A}$ and a canonical bipartite state Ω_A whose marginal χ_A has full support and whose steering map

$$\widehat{\Omega}_A : E^+_{\bar{A}} \to V^+_A, \qquad e \mapsto (I \otimes e)\Omega_A,$$

is cone-surjective: every positive state ray or every decomposition relevant to the theory can be induced by conditioning on the reciprocal side. Purity of Ω_A is desirable for the "one relational whole" interpretation but is not required for every algebraic consequence.

Wilce's conjugate-system reconstruction provides an important published analogue: a system is paired with an isomorphic conjugate and a bipartite state that uniformly and perfectly correlates corresponding tests; together with additional assumptions, this leads to Jordan structure [14].

6.2 Dynamic faithfulness and horizon extensionality

Cone-surjectivity plus full support already gives a form of dynamic faithfulness. If

$$(\mathcal{M} \otimes I)\Omega = (\mathcal{N} \otimes I)\Omega,$$

then their action on all steered states agrees, hence $\mathcal{M} = \mathcal{N}$ on the generated state space.

Add *Horizon Extensionality* (HE), injectivity of $\widehat{\Omega}_A$. Then $\widehat{\Omega}_A$ is an order isomorphism between the reciprocal effect cone and the state cone. Under a reciprocal symmetry identifying A with $\bar{A}$, one obtains a weak self-duality pairing such as

$$\langle a, b \rangle_\Omega = \Omega(a, \bar{b}).$$

This is close in spirit to Wilce's conjugate-based self-duality arguments. However, RHC+HE does not imply homogeneity or a rich enough physical transformation group.

7 The Homogeneity Detour: HM, RSM, Spectrality, and Recursive Faces

7.1 Horizon Mobility

Horizon Mobility (HM) asks that every interior state ρ be reachable from a maximally open state χ by a probabilistically reversible order automorphism:

$$F_\rho(\chi) \propto \rho.$$

Deterministic reversible channels are usually too restrictive because they often fix χ. The relevant transformations are reversible filters.

7.2 Reversible Salience Modulation

For a sharp frame $F = \{(\alpha_i, e_i)\}$ and positive weights $\mathbf{t} = (t_i)$, *Reversible Salience Modulation* (RSM) postulates an order automorphism $D_{F,\mathbf{t}}$ satisfying

$$D^*(e_i) = t_i e_i, \qquad D\alpha_i = t_i \alpha_i,$$

with multiplicative coherence

$$D_{\mathbf{s}} D_{\mathbf{t}} \sim D_{\mathbf{st}}.$$

If every interior state is spectral,

$$\rho = \sum_i p_i \alpha_i,$$

and χ is uniform on the frame,

$$\chi = \frac{1}{r} \sum_i \alpha_i,$$

then choosing $t_i = rp_i$ gives $D\chi = \rho$. Hence

$$\text{spectrality} + \text{uniformity} + RSM \Longrightarrow HM.$$

7.3 From Complete Sharp Articulation to recordable refinement

Complete Sharp Articulation (CSA) was the direct spectrality axiom: every state decomposes into a single perfectly distinguishable pure sharp frame. This was weakened to *Complete Recordable Articulation* (CRA): every state is the marginal of a joint state carrying a classical/sharp record perfectly correlated with a sharp test. Under sharpness, conditioning on a record outcome identifies the corresponding pure state, so CRA yields state-side spectrality.

A still deeper attempt used *Stable Recordable Refinement* (SRR), in which an articulation recursively refines into recorded branches. A key audit corrected an implicit mistake: memory of an earlier outcome does not imply that the physical state remains in the corresponding face. In quantum mechanics, one may record $\sigma_z = +$ and later measure σ_x; the record survives while z-fulfillment does not.

This required *Fulfillment-Preserving Refinement* (FPR), a nondemolition condition. If P_i projects/restricts to a branch face and T_{ij} is a refinement within that branch, then

$$P_i T_{ij} = T_{ij} = T_{ij} P_i.$$

To ensure refinement terminates, *Hereditary Sharp Refinability* (HSR) says every non-atomic sharp face admits a proper sharp refinement and inherits the same recursive system structure. Finite rank then yields atomic termination.

7.4 Hereditary Refinement and Lifting

These ideas were packaged into *Hereditary Refinement and Lifting* (HRL):

HRL0 sharp faces/corners are legitimate lower-rank systems;

HRL1 reversible symmetries of a residual system lift to the parent;

HRL2 arbitrary residual processes lift to the parent.

Wilce's subspace property is a published close analogue. In his Appendix B, lower-rank models obtained by excluding an outcome are again systems; symmetry lifting plus a weak global symmetry yields full symmetry by induction, while a stronger subspace property lifts more general processes and yields reversible filters [14].

Because global strong HRL can erase nontrivial superselection structure, the dialogue moved to a sectorial version: recursive closure and lifting are required only within coherent sectors.

8 Residual Autonomy and the Coherence-Extension Problem

8.1 Corners rather than tensor factors

A fulfilled sharp proposition need not leave a tensor-factor residual system. In quantum theory a projection P gives a corner $P\mathcal{H}$, not generally a factorization. Thus residual structure should be modeled by faces/corners.

If an idempotent restriction P_N splits as

$$A_N \xrightarrow{\iota_N} A \xrightarrow{\pi_N} A_N, \qquad \pi_N \iota_N = I, \qquad \iota_N \pi_N = P_N,$$

then any internal residual process T has a destructive extension

$$\widetilde{T} = \iota_N T \pi_N.$$

So mere process lifting is easy. What is substantive is *Reversible Residual Autonomy* (RRA): the restriction map from ambient automorphisms fixing the context onto residual automorphisms is surjective.

8.2 The coherence bridge

For a Hilbert decomposition

$$\mathcal{H} = P\mathcal{H} \oplus P^{\perp}\mathcal{H},$$

an internal unitary U extends as $U \oplus I$. But this extension also acts on off-diagonal coherence terms. In an EJA, the analogous structure is the Peirce decomposition

$$J = J_1(p) \oplus J_{1/2}(p) \oplus J_0(p),$$

where $J_{1/2}(p)$ is a coherence bridge. Thus lifting an internal symmetry is really a coherence-extension problem.

A general no-go observation followed: state/effect/RHC kinematics can be held fixed while the allowed transformation set is arbitrarily restricted. Therefore no purely kinematic principle can imply a rich dynamics unless some dynamical-realizability assumption is added.

8.3 The Jordan benchmark and geometric-mean scaling

EJAs show exactly how radial extensions should act. For a Jordan frame $\{e_i\}$, let

$$a = \sum_{i \le k} \sqrt{t_i}\, e_i, \qquad b = a + (u - p),$$

where $p = \sum_{i \le k} e_i$. The quadratic representation

$$P(b) = 2L(b)^2 - L(b^2)$$

is an invertible cone automorphism when b is positive and invertible. On Peirce components,

$$P(b)|_{J_{ii}} = c_i^2 I, \qquad P(b)|_{J_{ij}} = c_i c_j I.$$

Taking $c_i = \sqrt{t_i}$ gives

$$e_i \mapsto t_i e_i, \qquad z_{ij} \mapsto \sqrt{t_i t_j}\, z_{ij}.$$

Thus coherences scale by the geometric mean of endpoint salience weights. This is a striking Jordan analogue of amplitude-vs-probability square roots.

At this point, however, a major simplification became available: perhaps radial filters and homogeneity should not be primitive at all.

9 The Barnum–Hilgert Simplification

Barnum and Hilgert prove that a finite-dimensional compact convex state space is spectral and strongly symmetric if and only if it is affinely isomorphic to a simplex or the normalized state space of a simple Euclidean Jordan algebra [15].

Here *spectral* means every state has a convex decomposition into perfectly distinguishable pure states. *Strong symmetry* means that, for every N, the affine automorphism group acts transitively on ordered lists of N perfectly distinguishable pure states.

This theorem changed the strategy. Rather than postulating RSM/HM/self-duality/homogeneity and laboriously controlling coherence, one can aim to derive just:

$$\boxed{\text{spectrality} + \text{strong symmetry.}}$$

Jordan quadratic representations, homogeneous cones, and radial filters then become consequences of the resulting EJA structure rather than assumptions.

The theorem does *not* select complex quantum theory. Simple EJAs include real, complex, and quaternionic matrix algebras, spin factors, and the exceptional Albert algebra; simplices are the classical cases. Composite assumptions such as local tomography can further restrict the possibilities; Barnum and Wilce show how Jordan single-system structure plus suitable composites and a qubit can return complex finite-dimensional quantum mechanics with superselection rules [16].

10 Co-Horizon Extensibility and Hereditary Symmetry of Standing

10.1 The correct residual object

An important correction concerned what is held fixed. If x is an atomic outcome, the relevant residual system is the *co-horizon*

$$F_x = \{\rho : \rho(x) = 0\},$$

not "the structure inside a fulfilled atomic branch." In quantum theory, F_x corresponds to states on the orthogonal complement $x^\perp$.

Wilce's subspace property makes this lower-rank model part of the theory and requires symmetries of it to extend to the parent while fixing x [14]. This is almost exactly the *Coherent Angular Extensibility* (CAE) principle proposed in the dialogue.

If

$$G(A)_x = \{g \in G(A) : gx = x\},$$

and

$$r_x : G(A)_x \to G(A_x)$$

is restriction, full CAE is surjectivity of r_x.

A weaker form, *Frame CAE* (F-CAE), only requires the image to act transitively on ordered maximal frames of the residual system. Together with transitivity on individual atomic outcomes, this is enough for an induction to full frame symmetry. This is the same architecture as Wilce's lemma that symmetry plus the subspace property yields full symmetry [14].

10.2 Hereditary Symmetry of Standing

The phenomenological reformulation was *Hereditary Symmetry of Standing* (HSS). For a compatible partial frame

$$C = (x_1, \dots, x_k),$$

let

$$X_C = \{y : (x_1, \dots, x_k, y) \text{ is jointly distinguishable}\}$$

and let the pointwise stabilizer be

$$G_C = \{g : gx_i = x_i \ \forall i\}.$$

Then

$$\boxed{G_C \text{ acts transitively on } X_C.}$$

The intended reading is: fixing some compatible determinations does not create an arbitrary inequality of standing among the remaining alternatives.

Theorem 10.1 (HSS–strong-symmetry equivalence)**.** *In a finite-rank setting where distinguishable lists can be extended as needed, HSS for every partial distinguishable list is equivalent to transitivity on every ordered distinguishable list.*

Proof sketch. HSS with the empty context gives pure-state transitivity. Given two ordered k-frames, map the first element to the first target. At stage j, the already matched target prefix $C = (y_1, \ldots, y_{j-1})$ is fixed. The current image of x_j and y_j both lie in X_C, so HSS supplies a stabilizer element mapping one to the other without disturbing the prefix. Iteration maps the full ordered frame. Conversely, strong symmetry applied to (C, y) and (C, z) gives an automorphism fixing C and mapping y to z. □

This result shows that HSS is not mathematically weaker than strong symmetry; its value is explanatory. It decomposes a global group-transitivity axiom into the recursive claim that symmetry of standing survives conditioning.

10.3 Why HSS is not kinematically forced

Pure-state transitivity alone does not imply HSS. Take a classical trit whose full kinematic symmetry is S_3 but restrict the physical reversible group to the cyclic subgroup C_3. Globally the three pure states are transitive, but after fixing one point the residual two alternatives cannot be swapped by the physical stabilizer. Likewise, RHC can be kept fixed while reversible transformations are deleted. Kinematics does not determine dynamical richness.

This motivated the explicit separation of kinematic and dynamical symmetry.

11 NUDA: No Ungrounded Dynamical Asymmetry

11.1 Non-circular formulation

Fix a conditioning context C. Let $\mathscr{H}_C$ denote the complete *kinematic* conditioned horizon: state/effect structure, sharp contexts, refinement, reciprocal pairings, and independently specified background marks, but no list of allowed reversible transformations. Let

$$\mathcal{K}_C := \mathrm{Aut}_{\mathrm{kin}}(\mathscr{H}_C)$$

be the automorphism group of this structure. Let G_C be the actual physical reversible stabilizer. Necessarily

$$G_C \leq \mathcal{K}_C.$$

Two residual pure alternatives have the same kinematic constitutive profile iff they lie in the same $\mathcal{K}_C$-orbit.

Definition 11.1 (NUDA)**.** *No Ungrounded Dynamical Asymmetry requires*

$$y \sim_{\mathcal{K}_C} z \quad \Longrightarrow \quad \exists g \in G_C \text{ with } gy = z.$$

Equivalently, G_C and $\mathcal{K}_C$ induce the same orbit partition on residual pure alternatives.

NUDA is weaker than full automorphism realizability: it does not require every $\phi \in \mathcal{K}_C$ to be a physical process, only that kinematically equivalent alternatives are not split into different physical reversible orbits.

Quantum theory illustrates the distinction. Formal state-space symmetries may include antiunitary maps, whereas physical reversible quantum channels are unitary conjugations. Yet the physical unitary group is still transitive on pure rays and on residual pure rays after a compatible frame is fixed.

11.2 Depth hierarchy

The hereditary aspect can fail only after several conditioning steps. For a classical tetrahedron, take kinematic group S_4 and physical group A_4. At depth zero, A_4 is transitive on four points. After fixing one point, the stabilizer $A_3 \simeq C_3$ remains transitive on the other three. After fixing two points, however, swapping the remaining pair is odd and unavailable. Thus NUDA can hold at depths 0 and 1 but fail at depth 2.

This gives a hierarchy $NUDA^{(k)}$ and shows why recursive stabilizers are more informative than global transitivity.

11.3 Grounded asymmetry is allowed

NUDA is always relative to a complete conditioning context. Different charges, external fields, boundary conditions, superselection labels, or conserved quantities may legitimately split orbits. These must be represented independently in the kinematic structure; one may not invent an invisible label merely after noticing that a physical transformation is absent. Otherwise NUDA would be unfalsifiable.

12 Can Recursive Constitution Produce Kinematic Symmetry?

12.1 RCA alone does not imply HKT

The next question was whether Recursive Constitutive Articulation (RCA) could by itself yield *Hereditary Kinematic Transitivity* (HKT), the kinematic counterpart of HSS.

A counterexample is immediate. Take a classical simplex with perfectly recursive sharp face structure, but mark one atom red and the others blue. Every sharp face remains a valid lower-rank simplex; recursive articulation works perfectly. Yet automorphisms preserving the mark cannot exchange the red atom with a blue atom. Therefore

$$\boxed{RCA \not\Longrightarrow HKT.}$$

Even isomorphic descendant horizons are insufficient: two isomorphic substructures can occupy inequivalent embeddings in a larger structure. RCA describes descendants; HKT concerns the complete pointed embedding $(\mathscr{H}_C, y)$.

12.2 Sectorial anonymity and no primitive positional labels

The missing intuition was "no primitive positional labels." If y, z have the same intrinsic atomic type, isomorphic recursive descendants, and identical relations to all independently marked backgrounds, then there should be no extra fact that one is "branch 7" and the other "branch 8." A strong version of this principle already approaches HKT, so the dialogue sought a more constructive route.

This produced *No Ungrounded Kinematic Asymmetry.*

13 NUKA and Constitutive Homogeneity

13.1 One-point homogeneity

Let $\Gamma_C(y)$ be a grounded finite constitutive profile of y relative to C: intrinsic type, refinement descendants, sharp relations, transition data, and independently marked backgrounds, but not the automorphism orbit itself.

Definition 13.1 (NUKA)**.** *No Ungrounded Kinematic Asymmetry requires that if y and z have the same complete grounded constitutive type over C, then there is a kinematic automorphism of $\mathscr{H}_C$ fixing C and mapping y to z.*

This is a one-point homogeneity-over-context property. A stronger model-theoretic notion is ultrahomogeneity: every finite partial isomorphism extends to an automorphism. NUKA only needs the one-point cases relevant to building distinguishable frames.

A central warning appeared here. If $\Gamma_C(y)$ is too weak (for example, only local valence data), globally asymmetric structures can have identical local profiles. If $\Gamma_C(y)$ is defined as the entire pointed parent $(\mathscr{H}_C, y)$, NUKA becomes tautological. The constitutive language must therefore be rich enough to record genuine finite relational invariants but not define automorphism orbits by fiat.

13.2 Constitutive Structuralism

NUKA and Constitutional Separatedness are not the same theorem. Separatedness is a uniqueness principle: no two pure wholes over identical pure local data. NUKA is an extension/homogeneity principle: no inequivalent positions with the same grounded constitutive type. They can nevertheless be viewed as two consequences of a broader anti-haecceitistic doctrine:

$$\boxed{\text{No Constitutive Haecceities.}}$$

One clause removes surplus object identity; the other removes surplus positional identity.

14 GCA: Grounded Constitutive Amalgamation

14.1 The initial principle and its quantum correction

Suppose

$$A \hookrightarrow B_1, \qquad A \hookrightarrow B_2$$

are two constitutive extensions. An amalgam is a larger D containing compatible copies of B_1, B_2 agreeing on A. It is *strong* if

$$j_1(B_1) \cap j_2(B_2) = j_1(A) = j_2(A),$$

so gluing introduces no ungrounded new identifications.

Unrestricted amalgamation is too strong for quantum-logical structures. Bruns and Harding show that orthomodular lattices do not have the amalgamation property in general, but they do have the *strong Boolean amalgamation property*: two orthomodular lattices can be strongly amalgamated over a common Boolean subalgebra [17]. This prompted a quantum-safe correction.

Definition 14.1 (Context-Grounded Constitutive Amalgamation, CGCA)**.** *If two admissible contextual extensions agree on a common sharp/Boolean context C and contain no independently grounded contradiction, then they admit a strong amalgam in a larger contextual horizon. The amalgam need not make all new propositions jointly sharp or Boolean.*

Thus "coexistence in one horizon" is weaker than "simultaneous fulfillment." This is crucial for contextuality.

14.2 Compatibility with contextuality

Abramsky and Brandenburger characterize contextuality as an obstruction to the existence of a suitable global section over a measurement cover [18]. CGCA does not require such a section. It glues measurement/context structures into one larger horizon; it does not assign simultaneous values to every outcome in every context. The hierarchy is:

amalgamability of contexts $\supsetneq$ joint sharp compatibility $\supsetneq$ global noncontextual valuation (when it exists).

15 Horizon Saturation

15.1 Existential closure rather than unlimited plenitude

Naive saturation—"every mathematically consistent extension exists inside the same system"—is impossible for finite-capacity systems. A d-level system cannot internalize a configuration requiring $d+1$ mutually distinguishable outcomes. Saturation must be sector- and capacity-relative.

Fix a coherent sector type Σ. Let H be a complete horizon, $C \subset H$ a finite context, and $\tau(b/C)$ a finitely specified extension type. If there is an admissible same-sector enlargement $H \subset H'$ with

$$H' \models \exists b\, \tau(b/C),$$

then Horizon Saturation requires

$$H \models \exists b\, \tau(b/C).$$

This is a form of existential closure: a finite witness available in an admissible extension is already present internally.

Phenomenologically this was called *No Ungrounded Modal Exclusion*: a complete horizon has no arbitrary holes in its lawful finite possibility space.

15.2 CGCA plus saturation gives an internal extension property

CGCA and HSat perform distinct steps. Given an independently admissible extension B over $C \subset H$, CGCA embeds B together with H into a common enlargement H'. HSat then pulls the witness back into H. Thus

$$CGCA + HSat \Longrightarrow \text{internal one-point extension property.}$$

In a countable Fraisse setting, iterating such extensions back-and-forth yields homogeneity. In continuous structures, metric Fraisse theory replaces exact amalgamation by near amalgamation and produces approximate ultrahomogeneity [19]. In finite-dimensional compact settings, approximate orbit equivalence can often be upgraded to exact orbit equivalence by compactness of the relevant automorphism group.

16 Independence of CGCA and Horizon Saturation

The dialogue explicitly tested whether the two principles collapse. They do not.

Countermodel 16.1 (CGCA without HSat)**.** *Take the class of finite/simple graphs. It has free (hence strong) amalgamation: two graphs over a common induced subgraph can be united without adding cross-edges. Now choose as a particular horizon H an infinite edgeless graph. An extension H' can add a new vertex b adjacent to some $a \in H$. Then H' satisfies $\exists x\, E(a, x)$ while H does not. Thus the class has excellent amalgamation but this chosen horizon is not existentially closed.*

Therefore

$$CGCA \not\Longrightarrow HSat.$$

Countermodel 16.2 (HSat without strong CGCA)**.** *Let the admissible class consist of equivalence relations whose classes have size at most two. Let H be a countable union of two-element classes. Relative to this class, H is existentially rich: any finite pattern allowed in an extension over finitely many parameters can be reproduced using fresh existing pairs. However, from a one-point base $A = \{a\}$ consider two extensions $B_1 = \{a, b\}$ and $B_2 = \{a, c\}$ with aEb and aEc. Any common model with classes of size at most two must identify $b = c$. Hence ordinary amalgamation is possible only by an ungrounded identification, and strong amalgamation fails.*

Therefore

$$HSat \not\Longrightarrow CGCA.$$

The conceptual distinction is sharp:

$CGCA$	$HSat$
Can alternative branches coexist coherently?	Are admissible witnesses internally present?
branch coherence	modal plenitude
class/age property	completed-object property

17 Constitutive Modal Completeness

The independent principles can be unified by a genuine common strengthening rather than by deriving one from the other.

Definition 17.1 (Contextual Constitutive Modal Completeness, CCMC)**.** *Fix a coherent sector Σ and a complete horizon H_Σ. For every finitely generated realized constitutive subdiagram $A \hookrightarrow H_\Sigma$ and every finite/finitely generated extension diagram $A \hookrightarrow B$ that is jointly lawful for Σ, there exists a strong embedding*

$$B \hookrightarrow_A H_\Sigma.$$

Distinct elements required to remain distinct in the admissible diagram remain distinct in the realization.

CCMC says every finitely lawful pattern of further constitution is already represented in the complete horizon. It subsumes the functions of CGCA and HSat:

$$CCMC \Longrightarrow CGCA + HSat$$

for the relevant finite diagrams.

Several qualifications are essential.

1. **Sector/capacity preservation.** A finite diagram requesting $r+1$ mutually distinguishable atoms in rank r is not admissible.
2. **Whole-diagram admissibility.** Pairwise compatibility is not enough; the complete finite pattern must satisfy all higher consistency constraints.
3. **Contextuality preservation.** Co-realizing contexts in one horizon does not imply a global valuation across them.
4. **No unlimited fresh copying.** Two distinct witnesses are required only if the entire finite diagram demanding two witnesses is itself admissible.

A one-generator form is often enough: every admissible extension $A \subset \langle A, b\rangle$ can be realized over any copy of A in the complete horizon. Iterating one-generator extensions yields finite CCMC under suitable closure conditions.

CCMC is close mathematically to an injectivity/extension property and hence to homogeneity. Its foundational advantage is explanatory: it mentions no automorphism group. Symmetry emerges from repeatedly solving finite realization problems.

18 The Pre-Jordan Admissibility Kernel

The word "admissible" must not smuggle the desired Jordan geometry into the premises. The dialogue therefore proposed a deliberately permissive kernel that allows classical, Jordan, and many post-quantum GPTs.

A finitely generated constitutive diagram contains named states, effects, atomic sharp outcomes, contexts, and their finite operational relations. The proposed *Pre-Jordan Admissibility Kernel* (PJAK) has five object-level clauses plus strong embeddings.

18.1 PJAK clauses

OCC: Operational Convex Coherence. The named state/effect data admit a finite-dimensional ordered-vector-space realization with

$$0 \leq e(\omega) \leq 1, \qquad \sum_{e \in E} e = u$$

for specified tests, respecting all stated affine/convex relations.

COC: Contextual Overlap Coherence. If an effect occurs in multiple contexts, it is the same effect with the same probabilities; coarse-graining is additive and overlaps agree. No global joint refinement is required.

SAC: Sharp Atomic Consistency. Each atomic sharp outcome x has a unique normalized fulfilling state δ_x with

$$p(\delta_x, x) = 1.$$

For outcomes in the same sharp test,

$$p(\delta_x, y) = \delta_{xy}.$$

The uniqueness condition makes δ_x pure.

SFC: Sectorial Finite Capacity. Each diagram belongs to a fixed sector/type Σ with finite information capacity r_Σ and fixed independently grounded background invariants. No admissible diagram violates these.

HRC: Hereditary Restriction Consistency. Restricting a diagram to a sharp face/context gives another admissible diagram, and nested restrictions compose consistently.

Embeddings are required to be injective/strong and preserve all named structure, so gluing cannot collapse distinct possibilities without an explicit constitutive reason.

18.2 What PJAK deliberately does not assume

PJAK does *not* require self-duality, homogeneity, symmetric transition probabilities, a Jordan product, a Hilbert space, Gram positivity specific to quantum mechanics, purification, local tomography, reversible filters, or absence of higher-order interference. Square gbits and regular polygon state spaces can satisfy much of the kernel. This is a feature: post-quantum models should be eliminated by the substantive reconstruction principles rather than by hidden quantum assumptions in the definition of consistency.

Barnum–Hilgert make this economy especially useful: no separate "no higher-order interference" axiom is needed once spectrality and strong symmetry are obtained [15].

19 A Metric Constitutive Language $\mathcal{L}_{\text{mon}}$

To turn CCMC into a metric-Fraisse extension property, the dialogue specified a pre-Jordan multi-sorted language.

19.1 Sorts and primitive pairing

There are three primary sorts:

$$\mathbf{S} \quad \text{(normalized states)}, \qquad \mathbf{E} \quad \text{(effects)}, \qquad \mathbf{X} \quad \text{(atomic sharp outcomes)}.$$

There are maps

$$\delta : \mathbf{X} \to \mathbf{S}, \qquad \epsilon : \mathbf{X} \to \mathbf{E},$$

and the fundamental probability pairing

$$p : \mathbf{S} \times \mathbf{E} \to [0, 1].$$

Distinguished effect constants are 0 and u, with

$$p(\omega, 0) = 0, \qquad p(\omega, u) = 1.$$

State and effect extensionality require that probabilities separate operationally distinct states and effects.

19.2 Convex operations

For every rational $q \in [0, 1] \cap \mathbb{Q}$, include barycentric operations

$$m_q^S(\omega, \sigma) = q\omega + (1 - q)\sigma,$$

$$m_q^E(e, f) = qe + (1 - q)f,$$

with the usual identities and separate affinity of p. Preserving these operations forces any continuous automorphism of the structure to act affinely on the completed state space.

19.3 Metrics

The natural operational metrics would be

$$d_S(\omega, \sigma) = \sup_e |p(\omega, e) - p(\sigma, e)|,$$

$$d_E(e, f) = \sup_\omega |p(\omega, e) - p(\omega, f)|.$$

However, these suprema can increase when a finite structure is extended by new states/effects, which is awkward for Fraisse embeddings. The Fraisse-ready version therefore takes d_S, d_E, d_X as primitive bounded metrics and requires p to be Lipschitz. Operational-metric completeness can be imposed later on the saturated limit, recovering the supremum formulas.

19.4 Sharpness and contexts

A predicate Sharp(e) vanishes on sharp effects. Atomic outcomes satisfy

$$p(\delta_x, \epsilon_x) = 1,$$

with uniqueness of the normalized state achieving one.

For each $1 \leq k \leq r_\Sigma$, introduce a context predicate

$$T_k : \mathbf{X}^k \to [0, 1],$$

where $T_k(x_1, \ldots, x_k) = 0$ means the ordered tuple is a compatible sharp k-frame. If $T_k = 0$, then

$$p(\delta_{x_i}, \epsilon_{x_j}) = \delta_{ij}.$$

No global meet or universal compatibility operation is introduced.

19.5 Coarse-graining, refinement, and faces

Because effect addition is partial, use relations

$$\mathrm{Sum}_k(e_1, \ldots, e_k; e) = 0$$

to mean that e is their allowed coarse-grained sum, semantically enforcing

$$p(\omega, e) = \sum_i p(\omega, e_i).$$

A refinement predicate Ref$(e, f) = 0$ means fulfillment of e implies fulfillment of f. Sharp faces are definable:

$$F_e = \{\omega : p(\omega, e) = 1\}, \qquad F_e^{\perp} = \{\omega : p(\omega, e) = 0\}.$$

Thus RCA can be formulated without adding a separate face sort.

19.6 Grounded background predicates

A family of predicates B_j records real charge, species, boundary, spacetime, or superselection information. They may be included only when independently grounded; they may not be invented after the fact to encode automorphism orbits.

The basic Fraisse-ready signature is therefore

$$\mathcal{L}^F_{\mathrm{mon}} = \{\mathbf{S}, \mathbf{E}, \mathbf{X}; d_S, d_E, d_X; p; 0, u; m^S_q, m^E_q; \epsilon, \delta; \mathrm{Sharp}; T_k; \mathrm{Sum}_k; \mathrm{Ref}; B_j\}.$$

It contains no primitive inner product, Jordan product, amplitude, tensor product, purification map, reversible-transformation sort, self-dual identification, or Hilbert-space coordinate.

Because rational convex closure of finitely many generators is countably infinite, the finite objects of the eventual Fraisse category should technically be *finitely generated metric structures* or finite presentations rather than literally finite sets.

20 Sectorial Frame Anonymity and Pure-Sharp Completeness

Two small but important assumptions must be stated explicitly for the final theorem.

Definition 20.1 (Pure-Sharp Completeness, PSC)**.** *Every pure state of Ω_Σ is the unique fulfilling state δ_x of some atomic sharp outcome x.*

Without PSC, strong symmetry of the distinguished sharp skeleton need not imply strong symmetry of all pure states, which is what Barnum–Hilgert require.

Definition 20.2 (Sectorial Frame Anonymity, SFA)**.** *Any two ordered k-frames of perfectly distinguishable pure states in the same unmarked coherent sector generate isomorphic finitely generated constitutive substructures under the pointwise correspondence of frame elements.*

SFA is local anonymity, not global symmetry. It says two finite sharp frames carry no primitive labels distinguishing them. The existence of a global automorphism mapping them is to be produced by CCMC/homogeneity.

If genuine atomic labels exist, the theory should be decomposed into corresponding coherent sectors rather than forcing SFA globally.

21 Peeling RCA and Spectrality

The theorem-grade form of Recursive Constitutive Articulation is the following.

Definition 21.1 (Peeling RCA)**.** *Let F be any admissible sharp face of finite rank $m > 1$, and let $\omega \in F$ be normalized. There exist an atomic sharp outcome x of F, a number $\lambda \in [0,1]$, and a normalized state $\omega' \in F_x^{\perp}$ of rank at most $m-1$ such that*

$$\omega = \lambda\delta_x + (1-\lambda)\omega'.$$

Every sharp test of the residual co-face extends to a sharp test of F by adjoining x. Rank-one faces contain only their atomic pure state.

Theorem 21.2 (Recursive Spectrality)**.** *Peeling RCA implies that every normalized state is a convex combination of mutually perfectly distinguishable pure states.*

Proof. Induct on the face rank m. The $m = 1$ case is immediate. For $m > 1$, Peeling RCA gives

$$\omega = \lambda\delta_x + (1-\lambda)\omega',$$

with ω' in a lower-rank residual face. By induction,

$$\omega' = \sum_j q_j \delta_{y_j}$$

for a residual sharp frame $\{y_j\}$. Test-extension says $\{x\} \cup \{y_j\}$ is a sharp frame of F, so

$$\omega = \lambda\delta_x + \sum_j (1-\lambda) q_j \delta_{y_j}$$

is a spectral decomposition into jointly perfectly distinguishable pure states. □

Thus

$$\boxed{RCA \implies \text{spectrality.}}$$

This is the first decisive leg of the final reconstruction.

22 CCMC as a Fraisse-Style Homogeneity Principle

Let $\mathcal{K}_\Sigma$ be the class of finitely generated $\mathcal{L}^F_{\mathrm{mon}}$ structures satisfying PJAK and the independently specified sector theory T_Σ, with strong metric embeddings.

The exact finite extension property would say: for every $A \hookrightarrow B$ in $\mathcal{K}_\Sigma$ and every embedding $f : A \hookrightarrow H_\Sigma$, there is $g : B \hookrightarrow H_\Sigma$ with $g|_A = f$. Continuous structures often require the approximate form: for every $\varepsilon > 0$, $g|_A$ agrees with f to within ε.

Metric Fraisse theory shows that hereditary/joint-embedding/near-amalgamation classes have separable approximately ultrahomogeneous limits under suitable hypotheses [19]. Therefore the remaining technical task is not to assume homogeneity but to prove that the chosen PJAK class has the required near-amalgamation property and that CCMC identifies the intended complete horizon with the resulting generic/saturated object.

22.1 From CCMC and SFA to approximate strong symmetry

Take two ordered distinguishable k-frames

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k), \qquad \boldsymbol{\beta} = (\beta_1, \ldots, \beta_k).$$

SFA provides an isomorphism between their finitely generated constitutive substructures. Approximate ultrahomogeneity gives, for every $\varepsilon > 0$, a kinematic automorphism g_ε with

$$d(g_\varepsilon \alpha_i, \beta_i) < \varepsilon \qquad \forall i.$$

Thus the affine/kinematic automorphism group is approximately strongly symmetric.

22.2 Compact upgrade

In finite-dimensional compact state spaces, the relevant automorphism group is a closed subgroup of the isometry/affine group and is compact under a suitable uniform topology. If $\varepsilon_n \to 0$ and g_n approximately maps one frame to another, compactness yields a convergent subsequence $g_{n_j} \to g$. Continuity gives

$$g\alpha_i = \beta_i.$$

Hence approximate strong symmetry upgrades to exact strong symmetry.

Because automorphisms preserve all rational barycentric operations m_q, continuity implies that their action on Ω_Σ is affine. Therefore the strong symmetry obtained is exactly of the type used by Barnum–Hilgert.

23 The Provisional Monadic Jordan Reconstruction Theorem

We can now state the central theorem schema reached in the dialogue.

Theorem 23.1 (Monadic Jordan Reconstruction Theorem, conditional metric form)**.** *Let Σ be a finite-dimensional compact coherent sector with normalized state space Ω_Σ. Assume:*

1. *an $\mathcal{L}^F_{\mathrm{mon}}$ constitutive structure satisfying PJAK;*
2. *Pure-Sharp Completeness;*
3. *Sectorial Frame Anonymity;*
4. *Peeling Recursive Constitutive Articulation;*
5. *a metric CCMC/Fraisse extension property whose finitely generated age satisfies the hereditary, joint-embedding, near-amalgamation, separability, and compact-upgrade hypotheses described above.*

Then Ω_Σ is spectral and strongly symmetric under its affine automorphism group. Consequently, by Barnum–Hilgert, Ω_Σ is affinely isomorphic to a simplex or to the normalized state space of a finite-dimensional simple Euclidean Jordan algebra.

Proof architecture. Peeling RCA gives spectrality by the Recursive Spectrality theorem. SFA identifies any two ordered distinguishable frames as finite structures of the same constitutive type. CCMC plus the metric Fraisse hypotheses yields approximate ultrahomogeneity, hence approximate frame transitivity. Finite-dimensional compactness upgrades this to exact transitivity, and preservation of rational convex combinations makes the resulting automorphisms affine. Barnum–Hilgert then applies [15]. □

This is still a *conditional theorem schema* because the metric language/category and near-amalgamation theorem for PJAK have not yet been fully constructed. But the remaining gap is now sharply localized.

23.1 NUDA is not needed for the Jordan conclusion

A significant correction arose while formulating the theorem. Barnum–Hilgert use the affine automorphism group of the convex body. Thus the Jordan conclusion requires only kinematic affine strong symmetry, not that every symmetry be physically implementable.

NUDA is therefore not an axiom of the single-system Jordan theorem. Instead it yields a physical-dynamical corollary.

Corollary 23.2 (Dynamical completion)**.** *Let G_{phys} be the physically allowed reversible group of a sector satisfying the Monadic Jordan Reconstruction Theorem. If NUDA holds hereditarily on residual pure alternatives, then G_{phys} acts strongly symmetrically on distinguishable pure frames.*

Thus the mature separation is:

CCMC determines the geometry of possibilities; NUDA determines the richness of motion within it.

24 Where the Earlier Principles Now Sit

The exploratory dialogue introduced many principles that are no longer primitive in the final architecture. This is progress rather than wasted effort: each detour identified what later became derivable or unnecessary.

Principle	**Final role**
CRR	Early way to disfavor polygonal/PR models with discrete reversible symmetry; not part of the final theorem.
BCF / information causality	Recovers the Tsirelson radius on important slices but does not characterize the quantum correlator set.
RCC / JSR	EJA-level route to a state-dependent Gram representation and TLM correlator geometry; becomes natural once Jordan structure is already available.
Purification	Powerful published operational principle; judged too strong as a primitive monadological postulate.

Principle	Final role
PC, IPC, Constitutional Separatedness	Attempts to decompose Purity Preservation and ground pure product composition in anti-haecceitistic extensionality; mainly relevant to future composite-system theory.
RHC + HE	Candidate relational/composite principle yielding steering completeness, dynamic faithfulness, and weak self-duality; no longer needed for single-system Jordan reconstruction.
HM, RSM	Radial mobility/filter principles; expected to follow from Jordan quadratic representations rather than be primitive.
CSA / CRA	Direct or record-based routes to spectrality; superseded in the current single-system theorem by Peeling RCA.
SRR, FPR, HSR, HRL	Successive attempts to derive recursive spectrality and residual lifting while respecting nondemolition constraints; their sharp-face inheritance component survives in RCA/PJAK.
RRA / CAE / F-CAE	Dynamical/angular lifting principles; helped isolate the stabilizer structure later expressed as HSS and then split into NUKA/NUDA.
HSS	Recursive stabilizer formulation of strong symmetry; equivalent to strong symmetry under finite-rank assumptions.
NUKA	Kinematic one-point homogeneity; targeted as a consequence of CCMC rather than left primitive.
CGCA + HSat	Gluing and existential-closure ingredients; independent, but subsumed by the stronger CCMC extension principle.
NUDA	Irreducibly dynamical orbit-completeness principle; unnecessary for the kinematic EJA classification, necessary for physical realization of strong symmetry.

25 Relationship to Published Reconstruction Programs

The final architecture is best understood as a reorganization of themes already present in GPT reconstruction literature, combined with new monadological interpretations.

25.1 Information causality and the nonlocality boundary

Pawlowski et al. introduced information causality precisely to distinguish quantum from stronger no-signalling correlations [10]. The dialogue's BCF principle was a phenomenological reinterpretation of this constraint. The failure of BCF to characterize the complete quantum set reinforces the need for geometric principles.

25.2 Purification and sharp theories

Chiribella, D'Ariano and Perinotti show that purification has deep process-theoretic consequences, including reversible dilations and a Choi-like correspondence [11]. Chiribella and Scandolo show that

Causality, Purity Preservation, Purification, and Pure Sharpness yield diagonalization/spectrality [12]. The present program seeks a different route to spectrality because purification was judged ontologically too heavy for the monadological starting point.

25.3 Bit symmetry and self-duality

Muller and Ududec prove that bit symmetry—reversible equivalence of logical bits—implies self-duality in a GPT setting [13]. This supports the general idea that reversible symmetry can have deep geometric consequences, while the present program attempts to ground strong symmetry one level deeper through modal completeness rather than assume it as a dynamical primitive.

25.4 Wilce's conjugates, filters, and subspace property

Wilce's reconstruction is the closest published structural analogue to several stages of the dialogue. Conjugate systems and correlating bipartite states resemble RHC; reversible filters resemble RSM; the subspace property makes residual co-horizons into systems and lifts symmetries; a stronger subspace property lifts processes and supports filters [14]. The monadological development differs mainly in explanatory direction: it tries to replace symmetry-lifting assumptions with constitutive modal completeness and then recover symmetry as homogeneity.

25.5 Barnum–Hilgert as the decisive endpoint

Barnum–Hilgert provide the classification theorem that makes the final reduction possible [15]. Once spectrality and affine strong symmetry are established, no separate self-duality, homogeneity, or interference-order axiom is required to reach simple EJA state spaces and simplices.

25.6 Amalgamation and metric Fraisse theory

Bruns–Harding show why unrestricted quantum-logical amalgamation is too strong but Boolean-grounded strong amalgamation is viable [17]. Ben Yaacov's metric Fraisse theory supplies the mathematical paradigm in which near amalgamation and extension completeness generate approximate ultrahomogeneity [19]. The proposed CCMC route is explicitly inspired by this model-theoretic mechanism.

26 What Has Been Accomplished

The dialogue did not produce a completed reconstruction of quantum mechanics, but it produced a significant compression and clarification of the research problem.

1. **The superquantum problem was localized.** No-signalling and intersubjective consistency are too weak; PR correlations remain possible. Information-causality-type principles recover an important boundary but not the whole quantum set.

2. **The TLM geometry was connected to Jordan conjunction.** In an EJA, state-dependent inner products and a sharp-conjunction product rule give Gram correlators and the TLM inequality. This identifies the mathematical structure one ultimately wants to recover rather than assume.

3. **Several false or overstrong steps were removed.** Generic pure steering is not isometric; purification is too powerful to be an unanalyzed primitive; memory does not imply physical nondemolition persistence; recursive articulation does not imply symmetry; and kinematics cannot imply unrestricted dynamics.

4. **Spectrality was separated from symmetry.** Peeling RCA gives an explicit inductive route to spectrality. This mirrors the separation in Barnum–Hilgert rather than trying to derive all quantum structure from one monolithic axiom.

5. **Strong symmetry was given a recursive interpretation.** HSS reformulates strong symmetry as stabilizer transitivity after every compatible conditioning. This clarified the role of CAE/subspace-style lifting.

6. **Kinematic and dynamical symmetry were separated.** NUKA concerns the automorphism structure of possibilities; NUDA concerns the physical realization of kinematic orbits. The cyclic-trit and alternating-group counterexamples show the separation is substantive.

7. **NUKA was reframed as homogeneity rather than postulated symmetry.** Context-grounded amalgamation and horizon saturation suggest a Fraisse-style construction of homogeneous horizons.

8. **CGCA and HSat were proved independent by explicit abstract countermodels.** This justified replacing them, if desired, by the genuinely stronger CCMC rather than pretending one derived the other.

9. **A pre-Jordan consistency kernel was isolated.** PJAK is intentionally broad enough to admit post-quantum GPTs. This prevents the desired conclusion from being hidden in "admissibility."

10. **A concrete metric language was specified.** $\mathcal{L}^F_{\mathrm{mon}}$ is rich enough to express states, effects, convexity, sharp contexts, refinement, coarse-graining, and grounded backgrounds while containing no primitive Jordan or Hilbert structure.

11. **A precise theorem schema was reached.** PJAK + PSC + SFA + Peeling RCA + metric CCMC yields spectrality and affine strong symmetry, and therefore a simplex or simple EJA by Barnum–Hilgert, conditional on the unresolved near-amalgamation/Fraisse technical hypotheses.

27 Open Mathematical Problems

The remaining work is now unusually concrete.

27.1 Near amalgamation for PJAK structures

The central technical bottleneck is:

Conjecture 27.1 (PJAK Near-Amalgamation Conjecture)**.** *For an appropriately chosen sector theory T_Σ weaker than Jordan geometry, the finitely generated $\mathcal{L}^F_{\mathrm{mon}}$ models satisfying PJAK form a metric Fraisse class, or at least a class with the near-amalgamation and extension properties required for approximate homogeneity.*

This must be tested first in low-rank toy cases. Rank 2 is the natural laboratory; rank 3 is likely the first place where genuinely contextual overlap constraints become nontrivial.

27.2 Choosing the constitutive language without circularity

The language must encode every genuine finite invariant needed to distinguish configurations. In complex Hilbert space, pairwise transition probabilities alone may fail to encode phase-sensitive higher invariants; full finite Gram data suffice up to unitary gauge, but importing complex Gram matrices would be circular. The challenge is to identify a genuinely operational pre-Jordan replacement, potentially aided by reciprocal structures such as RHC.

27.3 Deriving Peeling RCA

Peeling RCA currently appears directly in the theorem. A deeper monadological derivation should start from hereditary sharp refinement, recordability, and fulfillment-preserving conditioning and prove the one-step convex decomposition plus test extension. This would complete the promised transition from phenomenological recursion to spectrality.

27.4 Composites and complex quantum theory

The single-system result stops at simple EJAs/simplices. A second-stage theory must address composites. RHC, Constitutional Separatedness, Pure Conditioning, and related reciprocity principles are natural candidates. Local tomography and suitable composite axioms are known to push Jordan theories toward ordinary complex quantum theory [16]. The exceptional Jordan algebra and superselection structure require special care.

27.5 Physical dynamics

NUDA remains a substantive law, not a consequence of kinematic state geometry. One should classify minimal forms of NUDA sufficient for operational tasks: full frame transitivity may be stronger than necessary if only certain experimental transformations must be physically available.

27.6 Infinite systems and QFT

Full saturation and compactness are finite-dimensional conveniences. In QFT one expects local/sectorial versions, with superselection sectors, boundary conditions, inequivalent representations, and global topological obstructions. The correct infinite-dimensional analogue may be local finite modal completeness rather than global saturation.

28 Philosophical Interpretation

The final reconstruction package is notable because it can be read in terms that remain recognizably monadological.

Recursive Constitution. A horizon is not a flat list of outcomes. It can be further articulated into determinate alternatives, and the residual horizon after an articulation remains a horizon of the same general kind. Mathematically, this drives spectral decomposition.

Constitutive Modal Completeness. A complete horizon is not arbitrarily sparse. Every finitely lawful contextual pattern of further constitution is represented somewhere within it. Mathematically, this drives extension richness and homogeneity.

No Ungrounded Dynamical Asymmetry. If the kinematic structure supplies no ground for distinguishing two residual possibilities, the actual reversible dynamics does not arbitrarily split them into inequivalent classes. Mathematically, this turns kinematic symmetry into physical symmetry.

The trilogy can also be expressed as three "no ungrounded" laws:

$$\boxed{\text{No ungrounded modal exclusion}}$$

$$\boxed{\text{No ungrounded kinematic asymmetry}}$$

$$\boxed{\text{No ungrounded dynamical asymmetry.}}$$

The first is internalized by CCMC, the second emerges as homogeneity, and the third remains a dynamical principle.

The striking possibility is that Jordan geometry may be characterized not by directly saying that state cones are homogeneous/self-dual, but by saying that finite perspectival possibilities recursively articulate and that a complete horizon contains all structurally lawful finite ways of doing so. This is still conjectural as a derivation from phenomenology, but it is mathematically meaningful enough to test.

29 Conclusion

The program began with a comparatively loose question: could a world of interacting Husserlian monads reproduce the actual world's quantum structure? It passed through classical three-monad models, Bell/CHSH nonlocality, GPTs, PR boxes, information causality, purification, steering, Euclidean Jordan algebra correlators, sharp refinement, residual faces, reversible filters, subspace properties, and model-theoretic homogeneity. The main achievement was not the accumulation of axioms but their progressive removal.

The current single-system core is surprisingly compact. A pre-Jordan operational language specifies only states, effects, sharp noemata, contexts, probabilities, convexity, refinement, and genuine background distinctions. Peeling Recursive Constitutive Articulation yields spectrality. Contextual Constitutive Modal Completeness, if realized as a metric Fraisse extension property, yields kinematic strong symmetry from local frame anonymity. Barnum–Hilgert then forces the finite-dimensional state space into the Jordan family. NUDA is separated off as a further law about which of those kinematic symmetries are physically realizable.

Thus the strongest present statement is not "we have derived quantum mechanics from Husserl." It is the more precise conditional claim:

$$\boxed{\text{recursive sharp constitution} + \text{finite contextual modal completeness} \Longrightarrow \text{spectral strongly symmetric geometry} =}$$

subject to the explicit model-theoretic and compactness hypotheses spelled out above.

The next decisive mathematical test is the near-amalgamation problem for finitely generated PJAK diagrams in $\mathcal{L}^F_{\mathrm{mon}}$. If that succeeds without importing Jordan structure into the sector theory, the program will have crossed an important threshold: the reconstruction will no longer be merely a philosophical reinterpretation of known axioms, but a new route from a generative theory of structured possibility to quantum-like convex geometry.

A Acronym and Principle Glossary

Acronym	**Meaning**
BCF	Bounded Communicative Fulfillment, the monadological information-causality principle.
CAE	Coherent Angular Extensibility, symmetry lifting from a co-horizon while fixed outcomes remain fixed.
CCMC	Contextual Constitutive Modal Completeness, finite internal realization of every jointly admissible contextual extension diagram.
CGCA	Context-Grounded Constitutive Amalgamation, strong gluing over a compatible sharp/Boolean base.
CRA	Complete Recordable Articulation, spectral decomposition mediated by a perfectly correlated record.
CRR	Continuous Reversible Refinement.
CSA	Complete Sharp Articulation, direct spectrality-style axiom.
FPR	Fulfillment-Preserving Refinement, nondemolition nesting of recursive refinement.
HE	Horizon Extensionality, injectivity of the reciprocal steering map.
HKT	Hereditary Kinematic Transitivity.

Acronym	Meaning
HM	Horizon Mobility, radial order-automorphism transitivity on interior states.
HRL	Hereditary Refinement and Lifting.
HSat	Horizon Saturation / existential closure of a complete horizon.
HSS	Hereditary Symmetry of Standing, recursive stabilizer transitivity.
IPC	Independent Pure Composition.
JSR	Joint Sharp Refinability.
NUDA	No Ungrounded Dynamical Asymmetry.
NUKA	No Ungrounded Kinematic Asymmetry.
OCC	Operational Convex Coherence, a PJAK clause.
PC	Pure Conditioning.
PDS	Pure/Constitutional Separatedness over fixed pure marginals.
PIE	Pure Intersubjective Extensionality.
PJAK	Pre-Jordan Admissibility Kernel.
PSC	Pure-Sharp Completeness.
RCA	Recursive Constitutive Articulation.
RCC	Reciprocal Conjunctive Constitution.
RHC	Reciprocal Horizon Completion.
RRA	Reversible Residual Autonomy.
RSM	Reversible Salience Modulation.
SAC	Sharp Atomic Consistency, a PJAK clause.
SFA	Sectorial Frame Anonymity.
SFC	Sectorial Finite-Capacity Consistency.
SRR	Stable Recordable Refinement.
TLM	Tsirelson–Landau–Masanes correlator geometry.

B Dependency Map

The final single-system reconstruction can be summarized schematically as

$$\begin{array}{ccc}
 & \text{PJAK} + \text{PSC} + \text{SFA} & \\
 \swarrow & & \searrow \\
\text{Peeling RCA} & & \text{CCMC} + \text{metric extension hypotheses} \\
\downarrow & & \downarrow \\
\text{Spectrality} & & \text{Affine strong symmetry} \\
 \searrow & & \swarrow \\
 & \boxed{\text{simple EJA state space or simplex}} & \\
 & \downarrow \quad (+\,NUDA) & \\
 & \text{physical reversible strong symmetry.} &
\end{array}$$

The older composite/relational branch remains largely orthogonal:

RHC + HE + composite principles $\rightsquigarrow$ steering reciprocity, state-effect duality, and composite geometry.

It is a natural candidate for the second stage after the single-system Jordan theorem.

C Selected Countermodels and Audits

1. **Generic pure steering is not isometric.** A nonmaximally Schmidt-weighted two-qubit pure state distorts transverse directions.
2. **BCF is not a full quantum-set characterization.** Some post-quantum correlators can pass simple information-causality/Uffink tests while violating TLM.
3. **No-signalling does not imply IPC.** Distinct pure relational states can share the same pure marginals, with a mixed bare product.
4. **RHC does not imply rich dynamics.** Keep state/effect/reciprocal kinematics fixed while deleting reversible transformations.
5. **Retention is not nondemolition persistence.** A recorded z outcome can coexist with later x measurement that destroys z fulfillment.
6. **RCA does not imply HKT.** A recursively perfect classical simplex with an intrinsic atom label has broken kinematic transitivity.
7. **Global transitivity does not imply hereditary transitivity.** A classical trit with physical group C_3 is globally transitive but has trivial point stabilizers.
8. **CGCA does not imply HSat.** Free-amalgamation graph class plus an edgeless non-existentially-closed horizon.
9. **HSat does not imply strong CGCA.** Equivalence classes of size at most two: an existentially rich union of pairs, but two distinct mate-extensions over one point cannot strongly amalgamate.

References

[1] *Possible Worlds of Interacting Husserlian Monads.* Unpublished research note from the monadological project, 2026.

[2] *Minimal Finite Models of Diachronic and Self-Conscious Overminds.* Unpublished technical note from the monadological project, 2026.

[3] *Wanting, Internal Constitutions, and Self-Authorship in Finite Husserlian Monads: From Proleptic Desire to Local Submonads and Diachronic Authorial Identity.* Self-contained research note in the monadological program, 27 August 2026.

[4] *Coherent Extrapolated Volition in Worlds of Interacting Husserlian Monads: Reflective Holonomy, Self-Grounding, Endogenous Social Choice, and Robust Action*. Self-contained research note in the monadological alignment program, 27 August 2026.

[5] B. S. Cirel'son (Tsirelson), "Quantum generalizations of Bell's inequality," *Letters in Mathematical Physics* 4, 93–100 (1980). doi:10.1007/BF00417500.

[6] S. Popescu and D. Rohrlich, "Quantum nonlocality as an axiom," *Foundations of Physics* 24(3), 379–385 (1994). doi:10.1007/BF02058098.

[7] J. Barrett, "Information processing in generalized probabilistic theories," *Physical Review A* 75, 032304 (2007). doi:10.1103/PhysRevA.75.032304.

[8] L. J. Landau, "On the violation of Bell's inequality in quantum theory," *Physics Letters A* 120(2), 54–56 (1987). doi:10.1016/0375-9601(87)90075-2.

[9] Ll. Masanes, "Necessary and sufficient condition for quantum-generated correlations," arXiv:quant-ph/0309137 (2003).

[10] M. Pawlowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Zukowski, "Information causality as a physical principle," *Nature* 461, 1101–1104 (2009). doi:10.1038/nature08400.

[11] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Probabilistic theories with purification," *Physical Review A* 81, 062348 (2010). doi:10.1103/PhysRevA.81.062348.

[12] G. Chiribella and C. M. Scandolo, "Operational axioms for diagonalizing states," *Electronic Proceedings in Theoretical Computer Science* 195, 96–115 (2015). doi:10.4204/EPTCS.195.8.

[13] M. P. Muller and C. Ududec, "Structure of reversible computation determines the self-duality of quantum theory," *Physical Review Letters* 108, 130401 (2012). doi:10.1103/PhysRevLett.108.130401.

[14] A. Wilce, "Conjugates, Filters and Quantum Mechanics," *Quantum* 3, 158 (2019). doi:10.22331/q-2019-07-08-158.

[15] H. Barnum and J. Hilgert, "Spectral Properties of Convex Bodies," *Journal of Lie Theory* 30, 315–344 (2020). doi:10.5802/jolt.1118. Preprint title: "Strongly symmetric spectral convex bodies are Jordan algebra state spaces," arXiv:1904.03753.

[16] H. Barnum and A. Wilce, "Local tomography and the Jordan structure of quantum theory," *Foundations of Physics* 44, 192–212 (2014). doi:10.1007/s10701-014-9777-1.

[17] G. Bruns and J. Harding, "Amalgamation of Ortholattices," *Order* 14, 193–209 (1998). doi:10.1023/A:1006090004568.

[18] S. Abramsky and A. Brandenburger, "The sheaf-theoretic structure of non-locality and contextuality," *New Journal of Physics* 13, 113036 (2011). doi:10.1088/1367-2630/13/11/113036.

[19] I. Ben Yaacov, "Fraisse limits of metric structures," *Journal of Symbolic Logic* 80(1), 100–115 (2015). arXiv:1203.4459.